RELATIVE LINEAR SETS AND SIMILARITY OF MATRICES WHOSE ELEMENTS BELONG TO A DIVISION ALGEBRA*

BY

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It is the purpose of this paper to develop the theory of the similarity transformation for matrices whose elements belong to a division algebra. In order to get a basis for generalization, the theory of the similarity transformation for matrices whose elements belong to a field is sketched in what seems to the authors a more suggestive method than those used heretofore.‡

L. A. Wolf's paper entitled *Similarity of matrices in which the elements are real quaternions*§ treats the case of the quaternion algebra over any subfield of the real field, by passing to the equivalent square matrices of order $2n$ with elements in the subfield.

Some of the results of the present paper are given in an abstract and a subsequent paper by N. Jacobson.|| Jacobson's results are to a certain extent more general. In the present paper a usable rational process is given for determining the equivalence or non-equivalence of matrices whose elements belong to a division algebra, and a theorem is developed concerning the rank of a polynomial in a matrix, which is not indicated by Jacobson. Some of the results contained herein were given by Ingraham at the summer meeting of the Society in 1935.¶

I. THE SIMILARITY TRANSFORMATION FOR THE COMMUTATIVE CASE

1. **Review of certain known theory.** If M is an $m \times n$ matrix (i.e., a matrix with m rows and n columns) with elements in a field F, the rank of M is

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‡ For somewhat similar treatment see F. Gantmacher, *La théorie géométrique des diviseurs élémentaires d'après M. Krull*, Transactions, University of Odessa, Mathematics 1, 1935, pp. 89–108 (Russian) avec Franz, resumé.

§ L. A. Wolf, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 737–743.

|| N. Jacobson, *On pseudo-linear transformations*, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 667–670. N. Jacobson, *Pseudo-linear transformations*, Annals of Mathematics, vol. 38 (1937), p. 485.

¶ M. H. Ingraham, *Characteristic spaces associated with the matrix whose elements belong to a division algebra*, Bulletin of the American Mathematical Society, vol. 41 (1935), p. 622.

equal to the maximum number of linearly independent rows and (which is the same) the maximum number of linearly independent columns. Also the rank of M is $n-s$ where s is the maximum number of linearly independent vectors ξ satisfying the equation $M\xi=0$. The equivalence of the two definitions of rank is essentially the theorem that the order of the linear extension of the columns of M plus the order of the maximal set right orthogonal to M is equal to the number of columns of M.

The set U of all vectors ξ such that $M\xi=0$ is a linear set over F. ξ and U are both said to be *right orthogonal* to M. Let $L(\xi_1, \xi_2, \cdots, \xi_n)$ denote the least linear set containing $\xi_1, \xi_2, \cdots, \xi_n$, i.e., the totality of all vectors of the form $\sum \xi_i a_i$ where the a_i are in the field F. $L(\xi_1, \xi_2, \cdots, \xi_n)$ is said to be the *linear extension* of $\xi_1, \xi_2, \cdots, \xi_n$.

2. **Definition of relative linear independence, relative linear extensions, and connection with similarity transformation.** Let M be an $n\times n$ square matrix with elements in a field F. A set of vectors $\xi_1, \xi_2, \cdots, \xi_k$ is defined to be *linearly independent relative to* M if no one of the ξ_i is the zero vector and if whenever a set of polynomials $h_1, h_2, \cdots, h_k$ with coefficients in F exists such that $\sum_{i=1}^{k} h_i(M)\xi_i=0$, it implies $h_i(M)\xi_i=0$, (i). A set U of vectors is said to be *linear relative to* M if (1) whenever ξ_1 is in U, $h(M)\xi$ is in U for every polynomial h in F; (2) if ξ_1 and ξ_2 are in U, then $\xi_1+\xi_2$ is in U. Denote by $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ the *linear extension of* $\xi_1, \xi_2, \cdots, \xi_k$ *relative to* M, i.e., the least set U containing $\xi_1, \xi_2, \cdots, \xi_k$ which is linear relative to M. It is the totality of vectors of the form $\sum_{i=1}^{k} h_i(M)\xi_i$, where the h_i are polynomials in F.

It should be noted that if $M=I$, these definitions reduce to the usual ones for linear independence, linear sets, and linear extensions.

The polynomial h and the vector ξ are said to be *associated relative to the matrix* M if $h(M)\xi=0$. As in the ordinary theory of the minimum equation of a matrix there is associated with every vector ξ relative to M a unique polynomial h of lowest degree with leading coefficient unity. This polynomial divides all others associated with the vector ξ and is said to be *minimally associated* with the vector ξ *relative to* M. Clearly, the minimum polynomial such that $h(M)=0$ is the least common multiple of all those associated with the various vectors ξ of the $n\times n$ vector space with elements in F.

Consider two similar matrices M and N. Then there exists a non-singular matrix T such that $T^{-1}MT=N$. Let $\xi_1, \xi_2, \cdots, \xi_k$ be linearly independent relative to M, and if h_i is minimally associated with ξ_i relative to M, then the vectors $\zeta_i=T^{-1}\xi_i$ are linearly independent relative to N, and h_i is the minimum polynomial associated with ζ_i relative to N.

Conversely, if M and N are two $n\times n$ matrices, if $\xi_1, \xi_2, \cdots, \xi_k$ is a set of vectors linearly independent relative to M, if $\zeta_1, \zeta_2, \cdots, \zeta_k$ is a set of vectors linearly independent relative to N such that $L_N(\zeta_1, \zeta_2, \cdots, \zeta_k) = L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the total vector space, and if $h_1, h_2, \cdots, h_k$ are minimally associated with $\xi_1, \xi_2, \cdots, \xi_k$ respectively relative to M and are also minimally associated with $\zeta_1, \zeta_2, \cdots, \zeta_k$ respectively relative to N, then M is similar to N. If $P=(\xi_1, M\xi_1, \cdots, M^{m_1-1}\xi_1, \xi_2, M\xi_2, \cdots, M^{m_2-1}\xi_2, \cdots, \xi_k, M\xi_k, \cdots, M^{m_k-1}\xi_k)$, and

$$Q = (\zeta_1, N\zeta_1, \cdots, N^{m_1-1}\zeta_1, \zeta_2, N\zeta_2, \cdots, N^{m_2-1}\zeta_2, \cdots, \zeta_k, \sqrt{\zeta_k}, \cdots, N^{m_k-1}\zeta_k), \tag{1}$$

where m_i is the degree of h_i, then P and Q are non-singular. If $T=PQ^{-1}$ then $T^{-1}MT=N$.

3. **Theory of relative linear independence and extension. Invariants of the similarity transformation.** If U is a linear set relative to M, and if $\xi_1-\xi_2$ is in U, then it is said that $\xi_1\equiv\xi_2 \bmod U$. The usual processes associated with congruences pertain to this definition. Moreover, given the vector ξ and U, there exists a minimum polynomial h with leading coefficient unity such that $h(M)\xi\equiv 0 \bmod U$ and this polynomial divides all other polynomials satisfying this congruence. The polynomial h is said to be *minimally associated* with the vector $\xi \bmod U$ *relative to* M. If h is associated with $\xi \bmod U$, then h is associated with $\xi \bmod U_1$ for all relative linear sets U_1 containing U.

The following five theorems are given for two vectors but clearly, by iteration, lead to obvious generalizations for more than two vectors. In all of these the equalities may be replaced by congruences with a relative linear set as modulus.

THEOREM 1. *If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M, and if h_1 and h_2 are relatively prime, then the vectors ξ_1 and ξ_2 are relatively linearly independent with respect to the matrix M.*

Let $g_1(M)\xi_1=g_2(M)\xi_2=\xi_3$. The polynomial minimally associated with ξ_3 must be a divisor of both h_1 and h_2 and is therefore 1. Hence if $g_1(M)\xi_1 = g_2(M)\xi_2$, then $g_1(M)\xi_1=g_2(M)\xi_2=0$, and the theorem is proved.

THEOREM 2. *If h_1 and h_2 are two relatively prime polynomials whose product $h=h_1h_2$ is minimally associated with the vector ξ relative to M, and if $\xi_1=h_2(M)\xi$ and $\xi_2=h_1(M)\xi$, then ξ_1 and ξ_2 are relatively linearly independent with respect to M and $L_M(\xi_1, \xi_2)=L_M(\xi)$. Moreover, the polynomials h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M.*

By Theorem 1 these vectors, ξ_1 and ξ_2, are relatively linearly independent with respect to M. Let $p_1h_1+p_2h_2=1$. Hence

$$\xi = p_2(M)h_2(M)\xi + p_1(M)h_1(M)\xi = p_2(M)\xi_1 + p_1(M)\xi_2.$$

Hence $L_M(\xi_1, \xi_2)$ contains $L_M(\xi)$. Obviously $L_M(\xi)$ contains $L_M(\xi_1, \xi_2)$, and therefore $L_M(\xi) = L_M(\xi_1, \xi_2)$.

By a similar method one can prove

THEOREM 3. *If h_1 and h_2 are relatively prime polynomials and are minimally associated with ξ_1 and ξ_2 respectively relative to M, then the polynomial minimally associated with $\xi_1+\xi_2$ relative to M is h_1h_2; moreover $L_M(\xi_1+\xi_2) = L_M(\xi_1, \xi_2)$.*

Suppose h_1 and h_2 are any two polynomials. One may express $h_1 = \prod p_i^{s_i}\prod q_i^{t_i}$ and $h_2 = \prod p_i^{v_i}\prod q_i^{u_i}$ where the p_i's and q_i's are polynomials which are relatively prime, pair by pair, and where $s_i \geqq v_i$ while $t_i < u_i$. If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2, then by Theorem 2 vectors ξ_3 and ξ_4 exist with which are minimally associated the polynomials $g_1 = \prod p_i^{s_i}$ and $g_2 = \prod q_i^{u_i}$ respectively, and by Theorem 3 the polynomial minimally associated with $\xi_3+\xi_4$ is g_1g_2, the least common multiple of h_1 and h_2. One may therefore state

THEOREM 4. *If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M, then there exists a vector ξ contained in $L_M(\xi_1, \xi_2)$ such that the least common multiple of h_1 and h_2 is the polynomial minimally associated with ξ relative to M.*

THEOREM 5. *If h_1 and h_2 are minimally associated with the vectors ξ_1 and ξ_2 respectively relative to M, if $h_1 = p_1h_2$, and if h_3 is minimally associated with $\xi_2 \bmod L_M(\xi_1)$ relative to M so that $h_3(M)\xi_2 = h_4(M)\xi_1$, then p_2 and p_3 exist such that $h_2 = p_2h_3$ and $h_4 = p_3h_3$; furthermore the vectors ξ_1 and $\xi_3 = \xi_2 - p_3(M)\xi_1$ are linearly independent relative to M, and h_3 is minimally associated with both ξ_3 and $\xi_3 \bmod L_M(\xi_1)$ relative to M.*

Since $h_2(M)\xi_2 = 0$, it follows that $h_2(M)\xi_2 \equiv 0 \bmod L_M(\xi_1)$. Hence p_2 exists such that $h_2 = p_2h_3$. Since $h_3(M)\xi_2 = h_4(M)\xi_1$, $0 = h_2(M)\xi_2 = p_2(M)h_3(M)\xi_2 = p_2(M)h_4(M)\xi_1$, and hence p_2h_4 must be divisible by h_1. Moreover $h_1 = p_1h_2 = p_1p_2h_3$, hence h_4 is divisible by p_1h_3 and hence by h_3, and there exists a p_3 such that $h_4 = p_3h_3$. Hence $h_3(M)\xi_2 = p_3(M)h_3(M)\xi_1$, and so if $\xi_3 = \xi_2 - p_3(M)\xi_1$, then $h_3(M)\xi_3 = 0$. Suppose a polynomial $h_5 \neq 0$ of lower degree than h_3 existed such that $h_5(M)\xi_3 = 0$, then $h_5(M)\xi_2 = h_5(M)p_3(M)\xi_1$, that is, $h_5(M)\xi_2 \equiv 0 \bmod L_M(\xi_1)$ where h_5 is of lower degree than h_3. Since this is impossible, h_3 is minimally associated with ξ_3 and with $\xi_3 \bmod L_M(\xi_1)$. From

this it can be proved at once that ξ_1 and ξ_3 are linearly independent relative to M.

From the discussion in the preceding part of this section it follows that given any set of vectors $(\zeta_1, \zeta_2, \cdots, \zeta_k)$, a set $(\xi_1, \xi_2, \cdots, \xi_m)$ may be found such that $\xi_1, \xi_2, \cdots, \xi_m$ are linearly independent relative to M and such that $L_M(\xi_1, \xi_2, \cdots, \xi_m) = L_M(\zeta_1, \zeta_2, \cdots, \zeta_k)$. Moreover by repeated use of Theorems 2 and 5 and their extensions to more than two vectors and to congruences this may be done in such a way that the polynomials h_i minimally associated with the ξ_i may be taken as powers of irreducible polynomials, or by repeated use of Theorems 4 and 5 and their extensions, the h_i may be chosen such that h_{i+1} divides h_i. The only limitations on the h_i are given by the following theorem.

THEOREM 6. *If $\xi_{11}, \xi_{12}, \cdots, \xi_{1s}$ and $\xi_{21}, \xi_{22}, \cdots, \xi_{2t}$ are two sets of vectors each of which is linearly independent relative to M such that $L_M(\xi_{11}, \xi_{12}, \cdots, \xi_{1s}) = L_M(\xi_{21}, \xi_{22}, \cdots, \xi_{2t})$, and if h_{1i} and h_{2i} are minimally associated with ξ_{1i} and ξ_{2i} respectively relative to M, and if h is an irreducible polynomial, and k any positive integer, then the number of the polynomials h_{1i} divisible by h^k is equal to the number of the polynomials h_{2i} divisible by h^k.*

Let $S_1 = L_M(\xi_{11}, \xi_{12}, \cdots, \xi_{1s})$ and $S_2 = L_M(\xi_{21}, \xi_{22}, \cdots, \xi_{2t})$. Since $S_1 = S_2$, the numbers of linearly independent vectors in S_1 and S_2 respectively which are orthogonal to $h(M)$ are equal. The number of linearly independent vectors of S_1 orthogonal to $h(M)$ is equal to the sum of the numbers of such vectors in the sets $L_M(\xi_{1i})$. If $h_{1i} = p_{1i}h$, then the m vectors $p_{1i}(M)\xi_{1i}$, $Mp_{1i}(M)\xi_{1i}, \cdots, M^{m-1}p_{1i}(M)\xi_{1i}$, where m is the degree of h, are linearly independent and form a base for the set of vectors orthogonal to $h(M)$ in $L_M(\xi_{1i})$. If h_{1i} is not divisible by h, then no vector of $L_M(\xi_{1i})$ other than the zero vector is orthogonal to $h(M)$. Hence the number of polynomials h_{1i} or h_{2i} divisible by h is n_1/m, where n_1 is the order of the set of vectors in S_1 (equal to the order in S_2) orthogonal to $h(M)$. This proves the theorem for $k=1$.

Let n_2 be the number of linearly independent vectors ξ in $L_M(h(M)\xi_{11}, \cdots, h(M)\xi_{1s}) = L_M(h(M)\xi_{21}, \cdots, h(M)\xi_{2t})$ which satisfy the equality $h(M)\xi = 0$, then by reasoning as above, n_2/m is the number of polynomials h_{1i} or the number of polynomials h_{2i} which are divisible by h^2, etc.

As an immediate consequence follows

THEOREM 7. *If $\xi_1, \xi_2, \cdots, \xi_k$ are linearly independent relative to M such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the total vector space, and if the polynomial h_i minimally associated with ξ_i relative to M is a power of an irreducible polynomial, then these h_i are the same for all sets of vectors satisfying these conditions.*

The polynomials of such a set are called the *characteristic divisors* of $M-\lambda I$, and if the irreducible polynomials are linear, the characteristic divisors of $M-\lambda I$ are identical with the elementary divisors as usually defined.

The invariance of the invariant factors also follows at once. The Dickson rational canonical form is a matrix N such that the h_i are the invariant factors when $\zeta_1, \zeta_2, \cdots, \zeta_k$ in Q of equation (1) are $\delta_1, \delta_{m_1+1}, \delta_{m_1+m_2+1}, \cdots, \delta_{m_1+m_2+\cdots+m_{k-1}+1}$, where the vector δ_j is the Kronecker delta vector, 1 in the jth place and zero elsewhere, and m_i is the degree of h_i. It is also easy to characterize similarly the Jordan canonical form based on elementary divisors, as well as forms based on the highest power of irreducible factors occurring in the invariant factors, namely, the characteristic divisors.

The above treatment has two distinct advantages. First, starting with any n linearly independent vectors, a set of vectors $\xi_1, \xi_2, \cdots, \xi_k$ linearly independent relative to M, such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the total vector space, may be found by rational means, usually without great effort. Secondly, all of the canonical forms are derived by the same theoretical development.

II. Outline of the problem for the case where the elements of the matrix belong to a quasi-field A

The chief reason that the extension of the above theory to the case where the elements of M belong to a quasi-field instead of a field is not trivial is that the rank of a polynomial in M is not invariant under the similarity transformation. Moreover, one cannot say that if $g(M)\xi=0$ then $g(T^{-1}MT)T^{-1}\xi=0$.

In order to obviate these difficulties, it was found that a new operation instead of ordinary multiplication should be defined. Given the polynomial $g(\lambda)=\sum\lambda^i a_i$, the matrix M, and the vector ξ, we define $g(M)\odot\xi$ to be the vector $\sum M^i\xi a_i$. In the case that all the elements involved belong to a field, $g(M)\odot\xi=g(M)\xi$. From this definition it follows that if $g(M)\odot\xi=0$, $g(T^{-1}MT)\odot T^{-1}\xi=0$. Whenever $g(M)\odot\xi=0$, ξ is said to be $\odot$-orthogonal (tentatively read as dot-orthogonal) to $g(M)$. It is seen that the order of the set of vectors $\odot$-orthogonal to $g(M)$ is an invariant of the similarity transformation. It does not follow if $g(M)\odot\xi=0$ that $g(M)\odot\xi a=0$, if a is a non-commutative element of A. Secondly, the operation $\odot$ cannot be performed upon ξ with only a knowledge of the matrix $g(M)$ but requires that one knows both M and the polynomial g. Moreover, if $g_1=\sum\lambda^i a_{1i}$, $g_2=\sum\lambda^i a_{2i}$, $g_1\odot g_2$ is defined to be $\sum_i\lambda^i g_2 a_{1i}=\sum_k\lambda^k\sum_{i+j=k}a_{2i}a_{1j}$. This last definition is nothing but the usual one for the product g_2g_1 of two polynomials in a commutative indeterminate with coefficients in A. If $g=g_1\odot g_2$, then g_2 is said to be an interior factor of g, and g_1 an exterior factor of g.

Dickson* has shown the existence of polynomials p_1 and p_2 such that $p_1 \odot g_1 + p_2 \odot g_2$ is the greatest common interior divisor, (g_1, g_2), of g_1 and g_2, and Ore† has proved the existence of the least common exterior multiple, $[g_1, g_2]$, of g_1 and g_2. The degree of $[g_1, g_2]$ is equal to the sum of the degrees of g_1 and g_2 less the degree of (g_1, g_2).

In a quasi-field A the total set of elements each of which is commutative with every other element of A is a field containing 0 and 1 and is called the *centrum* C.

THEOREM 8. *If f_1 and f_2 are two polynomials such that $f_1 \odot f_2 = h$, where h is a polynomial with coefficients in the centrum, then $f_1 \odot f_2 = h = f_2 \odot f_1$.*

Let $h = f_1 \odot f_2$, then $f_1 \odot f_2 \odot f_1 = h_1 \odot f_1 = f_1 \odot h$. Hence from the uniqueness of division, $h = f_2 \odot f_1$.

In terms of the $\odot$-process relative linear independence and the extension process L_M may readily be defined and it is not difficult to see that the theory of similarity transformation may be made to depend on the existence of sets of vectors $\xi_1, \xi_2, \cdots, \xi_k$ and $\eta_1, \eta_2, \cdots, \eta_k$ such that $L_M(\xi_1, \xi_2, \cdots, \xi_k) = L_N(\eta_1, \eta_2, \cdots, \eta_k)$ is the whole vector space, and such that both sets of vectors are minimally associated with the same set of polynomials, as in the last section.

III. VECTORS ORTHOGONAL TO POLYNOMIALS IN A MATRIX WITH ELEMENTS IN A FINITE DIVISION ALGEBRA A

A quasi-field A is a division algebra over the centrum C if there exists a finite number m of elements α_i in A such that every element a of A can be expressed as

$$a = \sum_{i=1}^{m} \alpha_i c_i,$$

where the coefficients c_i are in C.

If M is an $n \times n$ matrix and $\xi_1, \xi_2, \cdots$ a set of vectors with elements in a division algebra, the existence of a polynomial g such that $g(M) \odot \xi_i = 0$ can readily be established from the fact that not more than n of the vectors $M^j \xi_i$ $(j = 1, 2, \cdots)$ are right linearly independent. If $g(M) \odot \xi_i = 0$, then g and ξ_i are said to be *associated relative to* M. There is a unique polynomial g_i of lowest degree with leading coefficient unity associated with ξ_i. This g_i is an

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* L. E. Dickson, *Algebren und ihre Zahlentheorie*, Zurich, 1927, p. 256. The product $g_1 \odot g_2$ of this paper should be interpreted as $g_2 g_1$ of Dickson and interior factor as left-hand factor, etc.

† O. Ore, *Theory of non commutative polynomials*, Annals of Mathematics, (2), vol. 34 (1933), pp. 480–508.

interior factor of every other polynomial g associated with the vector ξ_i and it is said to be *minimally associated with ξ_i relative to M.*

A set of vectors is said to be *right linear* if for every a_1 and a_2 in A $\xi_1 a_1 + \xi_2 a_2$ is in the set whenever ξ_1 and ξ_2 are in the set. In particular, if a_1 and a_2 are in the centrum C, the set of vectors is said to be *linear relative to the centrum.* It follows that

THEOREM 9. *The maximal set of vectors S associated with the polynomial g relative to M is a linear set relative to the centrum.*

If $g = \sum \lambda^i a_i$ and a is any element of A, let $g_a = \sum \lambda^i a^{-1} a_i a$, then g_a will be called the transform of g by a.* In most applications a will be one of the basal elements α_i of A relative to its centrum. Clearly the transform of a $\odot$-product of two polynomials is the $\odot$-product of their transforms. Moreover, if ξ is associated with g, then ξa is associated with g_a, and if g is minimally associated with ξ, then g_a is minimally associated with ξa.

THEOREM 10. *If h is a polynomial with coefficients in the centrum, the maximal space associated with h relative to M is a right linear space; conversely, if S is any right linear space, the polynomial minimally associated with the entire space S relative to M has coefficients in the centrum.*

To prove the converse, let a be any element of A different from zero. $Sa = S$. Let $g = \sum \lambda^i a_i$ be the polynomial of lowest degree associated with every vector of S, hence $g(M) \odot S = 0$. Then $g_a(M) \odot S = g_a(M) \odot (Sa) = 0$. The degrees of g_a and g are equal, and the leading coefficient of each is unity. As the polynomial minimally associated with S is unique, $g_a = g$ and $a^{-1} a_i a = a_i$, (i). Hence every a_i is in the centrum C.

THEOREM 11. *If g is any polynomial with leading coefficient unity, there exists a matrix M and a vector ξ such that g is minimally associated with ξ relative to M.*

If

$$g = \lambda^k - \sum_{i=0}^{k-1} \lambda^i a_i,$$

consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 \cdots 0 & a_0 \\ 1 & 0 & 0 \cdots 0 & a_1 \\ 0 & 1 & 0 \cdots 0 & a_2 \\ \cdot & \cdot & \cdot \cdots \cdot & \cdot \\ 0 & 0 & 0 \cdots 1 & a_{k-1} \end{pmatrix} = (\delta_2, \delta_3, \cdots, \delta_k, \eta),$$

* This transform g_a is a special case of the transforms of a polynomial g by a second polynomial as used by Ore. In this paper, however, the only polynomials used for transforming others are constants.

where δ_i is the Kronecker δ and η is the kth column of M. Then δ_1 is effective as ξ.

THEOREM 12. *If g is any polynomial in the division algebra A, then the least common exterior multiple of g and its transforms by the basal elements α_i of A, i.e., $[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$, is a polynomial with coefficients in the centrum.*

From the preceding theorem there exists a matrix M and a vector ξ such that g is minimally associated with ξ relative to M. Any element in U, the right linear extension of ξ, is of the form $\xi(\sum\alpha_i c_i)$, where the c_i are in the centrum. Since g_{α_i} is minimally associated with $\xi\alpha_i$, $h=[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$ is minimally associated with U, and hence by Theorem 10, h has coefficients in the centrum.

If $h=[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$, then h is the polynomial of lowest degree with coefficients in the centrum which has g as an interior factor, and g is said to *define* h. If h is the polynomial of lowest degree with coefficients in the centrum which is associated with ξ relative to a matrix M, then h is said to be minimally associated with ξ over C relative to M. It follows that if h_1 is a polynomial in C and is associated with ξ and if h is minimally associated with ξ over C, then h is a factor of h_1. Also, if any polynomial g is a factor of h_1, and if g defines h, then h is a factor of h_1.

THEOREM 13. *If g is minimally associated with ξ relative to M, and if g defines h, then h is minimally associated with ξ over the centrum relative to M.*

THEOREM 14. *If g is an irreducible polynomial in A, then*

$$h=[g, g_{\alpha_2}, \cdots, g_{\alpha_m}]$$

is a polynomial irreducible in the centrum C.

Suppose $h=h_1\odot h_2$, where h_1 and h_2 are in C, then g divides either h_1 or h_2 for if this were not the case, polynomials p_1, q_1, p_2, q_2 exist such that $p_1\odot h_1=1+q_1\odot g$ and $p_2\odot h_2=1+q_2\odot g$, and hence

$$p_1 \odot h_1 \odot p_2 \odot h_2 = p_1 \odot p_2 \odot h_1 \odot h_2 = 1 + (q_1 + q_2 + q_1 \odot g \odot q_2) \odot g$$

which would necessitate g and $h_1\odot h_2$ being relatively prime, contrary to hypothesis.

THEOREM 15. *If g defines h but no interior factor of g of lower degree than the degree of g defines h, and if h is irreducible in C, then g is irreducible in A.*

THEOREM 16. *If g_1 defines h_1, if g_2 defines h_2, and if $g_1\odot g_2$ defines h, then h is a factor of $h_1\odot h_2$ and $[h_1, h_2]$ is a factor of h.*

Let $h_1=f_1\odot g_1$ and $h_2=f_2\odot g_2$. By Theorem 8

$$h_1 \odot h_2 = f_1 \odot g_1 \odot f_2 \odot g_2 = f_2 \odot f_1 \odot g_1 \odot g_2,$$

and hence h is a factor of $h_1 \odot h_2$. If $h = f \odot g_1 \odot g_2$, then also from Theorem 8, $[h_1, h_2]$ is a factor of h.

The following two theorems are corollaries of the above theorem.

THEOREM 17. *If $g = g_m \odot g_{m-1} \odot \cdots \odot g_1$ where the g_i are irreducible, and if $h_1, h_2, \cdots, h_t$ are distinct polynomials in C where s_j of the g_i each define h_j, $s_j > 0$, and if g defines h, then h divides $\prod h_j^{s_j}$ and is divisible by $\prod h_j$.*

THEOREM 18. *If g_1 defines h_1 and g_2 defines h_2, and if $(h_1, h_2) = 1$, then $g_1 \odot g_2$ defines $h_1 \odot h_2$.*

THEOREM 19. *If g_1 defines h_1, if h_2 divides h_1, and if $g_2 = (g_1, h_2)$, then g_2 defines h_2.*

Suppose that g_2 defines h_3. By hypothesis g_2 is an interior factor of h_2, hence h_3 is a factor of h_2. Let $h_2 = h_4 \odot h_3$, $h_1 = h_5 \odot h_2 = h_5 \odot h_4 \odot h_3$, and $g_1 = g_3 \odot g_2$. From Theorem 16, g_3 must define a multiple of $h_5 \odot h_4$, call it $h_6 \odot h_5 \odot h_4$. Let $h_6 \odot h_5 \odot h_4 = g_4 \odot g_3$. If h_4 is of degree greater than zero, g_3 and h_4 must have a common interior divisor other than 1, for if $(g_3, h_4) = 1$ then there exist polynomials p_1 and p_2 such that $p_1 \odot g_3 + p_2 \odot h_4 = 1$. Consequently, g_3 is an interior divisor of $h_6 \odot h_5$ which is impossible. Let g_5 be the common factor of g_3 and h_4. From this it follows that $g_5 \odot g_2$ is an interior factor of g_1 and of h_2, contrary to hypothesis.

If in Theorems 1–4 of parts I and II it is understood that all coefficients of the polynomials h_i are in the centrum, these theorems apply to the case in which the basic number system is a division algebra rather than a field. These same modifications apply to the proofs of these theorems except that the polynomials g_1 and g_2 in the proof of Theorem 1 may not have coefficients in the centrum. Furthermore, the polynomial h_i minimally associated with ξ_i relative to a matrix M is interpreted to be the polynomial with coefficients in the centrum minimally associated with ξ_i relative to M throughout these theorems.

THEOREM 20. *If g, any polynomial in A, defines h, and if h is associated with ξ over the centrum relative to M, then there exists a set of vectors ξ_i, where ξ_i is associated with g_{α_i}, such that $\xi = \sum_{i=1}^{m} \xi_i$.*

Let $h = f_{\alpha_i} \odot g_{\alpha_i}$. The polynomials $f, f_{\alpha_2}, \cdots, f_{\alpha_m}$ have no common exterior factor other than 1, for if there were such, let it be f_1. Then $h = f_1 \odot h_1$ where h_1 is a common multiple of g and its transforms and is of lower degree than h. Hence there exist polynomials p_i such that $\sum_i f_{\alpha_i} \odot p_i = 1$. Let $\xi_i = f_{\alpha_i}(M) \odot p_i(M) \odot \xi$ and the theorem follows at once. Moreover, $\xi_i \alpha_i^{-1}$ is associated with g.

THEOREM 21. *If U is a relative linear set and if h is minimally associated with a vector ξ* mod U *over the centrum C relative to M, and if g defines h, but no interior factor of g of lower degree than the degree of g defines h, then there exists a vector η in $L_M(\xi)$ such that*: (1) *g is minimally associated with η* mod U, *and* (2) *there exists a polynomial p such that $g(M)\odot\eta = p(M)\odot h(M)\odot\xi$.*

Consider $\xi_{1i}=\xi_i\alpha_i^{-1}$ where the ξ_i is defined as in the proof of Theorem 20. Let h_i be minimally associated with ξ_{1i} mod U over C. Since h, which is minimally associated with ξ mod U, is associated with each ξ_i mod U, h both divides and is divisible by $[h_1, h_2, \cdots, h_m]$, hence $[h_1, h_2, \cdots, h_m]=h$. For any polynomial q and vector ξ,

$$(q(M) \odot \xi)\alpha_i^{-1} = q_{\alpha_i^{-1}}(M) \odot (\xi\alpha_i^{-1}).$$

Also there exists a polynomial q_1 such that

$$q_{\alpha_i^{-1}}(M) \odot (\xi\alpha_i^{-1}) = q_1(M) \odot \xi.$$

Hence for some q_2,

$$\begin{aligned}\xi_{1i} &= (f_{\alpha_i}(M) \odot p_i(M) \odot \xi)\alpha_i^{-1} \\ &= f(M) \odot p_{i\alpha_i^{-1}}(M) \odot (\xi\alpha_i^{-1}) \\ &= f(M) \odot q_2(M) \odot \xi.\end{aligned}$$

Hence

$$g(M) \odot \xi_{1i} = h(M) \odot q_2(M) \odot \xi,$$

so that any relative linear combination η of the ξ_{1i} whose coefficients are polynomials in C will satisfy condition (2) of the theorem. The existence of such a combination η with which h is minimally associated is guaranteed by Theorem 4. This η will also satisfy condition (1).

THEOREM 22. *If g_1 and g_2 each defines h but no interior divisor of g_1 or of g_2 of degree less than the degree of g_1 or g_2 respectively defines h, then g_1 and g_2 are of the same degree.**

Let $\langle g\rangle$ represent the degree of any polynomial g. Suppose $\langle g_1\rangle \leqq \langle g_2\rangle$. By Theorem 11 a matrix M and a vector ξ_1 can be chosen such that the order of M is equal to $\langle g_1\rangle$ and g_1 is minimally associated with ξ_1 relative to M. By Theorem 21 the existence of a vector ξ_2 is established such that g_2 is minimally associated with ξ_2 relative to M, but the polynomial minimally associated with any vector relative to M is of degree not more than the order of M. Hence $\langle g_1\rangle = \langle g_2\rangle$.

Polynomials such as g_1 and g_2 of Theorem 22 are said to be *polynomials of minimum degree defining h*. The degree of g_1 and g_2 is dependent upon h

* This proof, independent of any theorem of unique factorization, seems of interest.

alone, and the degree of such polynomials shall be denoted by the symbol $\langle\langle h\rangle\rangle$, that is, $\langle\langle h\rangle\rangle = \langle g_1\rangle = \langle g_2\rangle$.

THEOREM 23. *If h is irreducible in C and if g is a polynomial of minimum degree defining h, then the degree of h is an integral multiple of the degree of g.*

THEOREM 24. *If g_1 defines h_1 and g_2 defines h_2, then there exists some transform $g_{2\alpha_i}$ of g_2 such that $g_{2\alpha_i} \odot g_1$ defines h_3 where h_1 is a factor of h_3 but $h_3 \neq h_1$ and h_3 is a factor of $h_2 \odot h_1$.*

Suppose for every i, $g_{2\alpha_i} \odot g_1$ defines h_1 and therefore is an interior factor of h_1. Then $[g_2 \odot g_1, g_{2\alpha_2} \odot g_1, \cdots, g_{2\alpha_m} \odot g_1] = h_2 \odot g_1$ is an interior factor of h_1, and therefore there exists a polynomial f such that $h_1 = f \odot h_2 \odot g_1 = h_2 \odot f \odot g_1$. Hence $f \odot g_1$ is in the centrum, which contradicts the hypothesis.

One can readily obtain

THEOREM 25. *If h is an irreducible polynomial and if g_1 is a polynomial of minimum degree defining h, there exists a set of transforms g_i $(i = 1, 2, \cdots, t)$ of g_1 by the basal elements α_i such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ is a polynomial of minimum degree defining h^t, and hence $\langle\langle h^t\rangle\rangle = t\langle\langle h\rangle\rangle$.*

THEOREM 26. *If $\xi_1, \xi_2, \cdots, \xi_k$ is a set of vectors linearly independent relative to M such that $L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the whole space, and if g_i is the minimum polynomial associated with ξ_i relative to M, then the rank of $h(M)$, where h is any polynomial in the centrum, is equal to $m - \sum_{i=1}^{k} \langle g_{1i}\rangle$ where m is the order of M and $g_{1i} = (h, g_i)_{ex}$, the greatest common exterior divisor of h and g.*

Since $g_{1i} = (h, g_i)_{ex}$, let $g_i = g_{1i} \odot g_{2i}$ and $h = g_{1i} \odot g_{3i} = g_{3i} \odot g_{1i}$. Hence $h(M) \odot g_{2i}(M) \odot \xi_i = 0$ and also $h(M) \odot M^j \odot g_{2i}(M) \odot \xi_i = 0$. Since g_i is the minimum polynomial associated with ξ_i, the number of vectors $M^j \odot g_{2i}(M) \odot \xi_i$ right linearly independent and orthogonal to $h(M)$ is equal to the degree of g_{1i}, $\langle g_{1i}\rangle$. Furthermore, since $\xi_1, \xi_2, \cdots, \xi_k$ are linearly independent relative to M by hypothesis, all vectors of the form $M^j \odot g_{2i}(M) \odot \xi_i$ for $i = 1, 2, \cdots, k$ and $j = 0, 1, 2, \cdots, \langle g_{1i}\rangle - 1$ are right linearly independent and orthogonal to $h(M)$. Consequently it follows that the rank of $h(M)$ is $m - \sum_{i=1}^{k} \langle g_{1i}\rangle$; for if any other vector ξ were orthogonal to $h(M)$ it would be dependent upon those orthogonal to $h(M)$ in each of $L_M(\xi_1), L_M(\xi_2), \cdots, L_M(\xi_k)$.

IV. EXISTENCE FOR n-SPACE OF A BASE, LINEARLY INDEPENDENT RELATIVE TO AN $n \times n$ MATRIX

If M is an $n \times n$ matrix, and if the minimum polynomial in C of M is $\prod h_i^{k_i}$, where the h_i are distinct irreducible polynomials, if a linearly independent base relative to M were found for the space associated with $h_i^{k_i}$, then a line-

arly independent base relative to M for the whole space is the totality of vectors in the bases for the separate spaces associated with each of the $h_i^{k_i}$, according to Theorem 2.

The rank of $\prod_{i\neq j} h_i^{k_i}(M)$ is equal to the order of the space orthogonal to $h_j^{k_j}(M)$. Hence the columns of $\prod_{i\neq j} h_i^{k_i}(M)$ form a base for the space associated with $h_j^{k_j}$. At least one of these columns is a vector such that $h_j^{k_j}$ is minimally associated with it over C relative to M or else h_j appears to a lower power than the k_jth in the minimum polynomial of M in C.

To simplify the notation, let h^{k_1} be the highest power of an irreducible polynomial h in C occurring in the minimum polynomial of M in C, and let g_1 be a polynomial of minimum degree defining h. By Theorem 24 there exist transforms, $g_{k_1}, g_{k_1-1}, \cdots, g_1$, of g_1 by certain basal elements α_i of A such that $g_{k_1}\odot g_{k_1-1}\odot \cdots \odot g_1$ is a polynomial of minimum degree defining h^{k_1}. By the preceding paragraph and Theorem 21 there exists a vector ξ_1 such that $g_{k_1}\odot g_{k_1-1}\odot \cdots \odot g_1$ is the minimum polynomial associated with ξ_1, and h^{k_1} is minimally associated over C with ξ_1. Since $h^t(M)\odot\xi_1$ and $g_t(M)\odot g_{t-1}(M)\odot \cdots \odot g_1(M)\odot \xi_1$ are minimally associated with $g_{k_1-t}\odot g_{k_1-t-1}\odot \cdots \odot g_1$ and $g_{k_1}\odot g_{k_1-1}\odot \cdots \odot g_{t+1}$ respectively, the order of $L_M(h^t(M)\odot\xi_1)$ is equal to the order of $L_M(g_t(M)\odot g_{t-1}(M)\odot \cdots \odot g_1(M)\odot\xi_1)$. These two spaces are identical since the first is contained in the second. Therefore, if f_1 is any polynomial such that $h^{k_1-t}(M)\odot f_1(M)\odot\xi_1=0$ there exist polynomials p_1 and p such that

$$(2)\qquad \begin{aligned} f_1(M)\odot\xi_1 &= p_1(M)\odot h^t(M)\odot\xi_1 \\ &= g_t(M)\odot g_{t-1}(M)\odot\cdots\odot g_1(M)\odot p(M)\odot\xi_1; \end{aligned}$$

i.e., any vector in $L_M(\xi_1)$ orthogonal to $h^{k_1-t}(M)$ is of the form given above.

Suppose $k_2\leqq k_1$ is the largest integer such that there exists a vector ξ_{21} in the space associated with h^{k_1} for which h^{k_2} is the minimum polynomial in C associated with ξ_{21} mod $L_M(\xi_1)$. This vector ξ_{21} may be found among the columns of $\prod_{i\neq j} h_i^{k_i}(M)$ where $h=h_j$. Since $f=g_{k_2}\odot g_{k_2-1}\odot\cdots\odot g_1$ is a polynomial of minimum degree defining h^{k_2}, by Theorem 21 the existence of a vector ξ_{22} is established such that f is minimally associated with ξ_{22} mod $L_M(\xi_1)$, and also the existence of p_2 is assured such that $f(M)\odot\xi_{22}=h^{k_2}(M)\odot p_2(M)\odot\xi_{21}$ and hence the vector $f(M)\odot\xi_{22}$ is in $L_M(\xi_1)$ and orthogonal to $h^{k_1-k_2}(M)$. If equation (2) is applied with $t=k_2$, there exists a polynomial p_3 such that $f(M)\odot\xi_{22}=f(M)\odot p_3(M)\odot\xi_1$. Take $\xi_2=\xi_{22}-p_3(M)\odot\xi_1$. It follows that $f(M)\odot\xi_2=0$, and since $\xi_2\equiv\xi_{22}$ mod $L_M(\xi_1)$, f is minimally associated both with ξ_2 and with ξ_2 mod $L_M(\xi_1)$.

Let k_3 be the largest integer such that there exists a vector ξ_{31} in the space associated with h^{k_1} such that h^{k_3} is minimally associated over C with

ξ_{31} mod $L_M(\xi_1, \xi_2)$. The polynomial h^{k_1} is associated with ξ_{31}, moreover h^{k_2} is associated with ξ_{31} mod $L_M(\xi_1)$ because of the maximal property of k_2. Then in $L_M(\xi_1, \xi_2, \xi_3)$ there will be a ξ_3 minimally associated with $g_{k_3} \odot g_{k_3-1} \odot \cdots \odot g_1$ mod $L_M(\xi_1, \xi_2)$ and hence minimally associated with h^{k_3} mod $L_M(\xi_1, \xi_2)$ over C, and such that $h^{k_3}(M)\xi_3 = 0$.

This process may be continued to complete the base for the total vector space.

It has been proved by Theorem 26 that

$$rk(h^k(M)) = m - \sum_i \langle (h^k, g_{k_i} \odot g_{k_i-1} \odot \cdots \odot g_1) \rangle.$$

Hence

$$rk(h^k(M)) = m - k\langle\langle h \rangle\rangle \text{ (number of } k_i \geqq k) - \sum_i k_i \langle\langle h \rangle\rangle$$

where the sum runs for values $k_i < k$. From this it follows that the polynomials associated with the above canonical proper base of the relative set equal to the total vector space are completely determined by a knowledge of the rank of the powers of h_i, where the h_i are the irreducible factors of the minimum equation of M. Since in part I it was shown that the matrices M and N are similar if relative to M and N there exists a proper base for the total vector space associated with the same set of minimum polynomials, Theorem 27 follows.

THEOREM 27. *If $h = \prod_{i=1}^{l} h_i^{t_i}$ is the minimum polynomial in the centrum for each of two matrices M and N, where h_i is irreducible, a necessary and sufficient condition that the matrix M is similar to the matrix N is that the rank of $h_i^j(M)$ is equal to the rank of $h_i^j(N)$ for every $i = 1, 2, \cdots, l$ and $j = 1, 2, \cdots, t_i$.*

If ξ_{ij}, $(i = 1, 2, \cdots)$, $(j = 1, 2, \cdots)$, is a set of linearly independent vectors whose linear extension relative to M is the whole space, if g_{ij} is the minimum polynomial associated with ξ_{ij}, and if g_{ij} is a polynomial of minimum degree defining $h_i^{k_{ij}}$, such an $h_i^{k_{ij}}$, where h_i is irreducible, is called a characteristic divisor of M. Hence the necessary and sufficient condition that two matrices be similar is that the characteristic divisors of the two matrices be equal. It is useful to note that the above implies that two matrices M and N are similar if for every polynomial h with coefficients in the centrum the rank of $h(M)$ is equal to the rank of $h(N)$.

Since two matrices are equivalent only if $rk(h^k(M)) = rk(h^k(N))$ whenever h is irreducible and k is a positive integer, a necessary condition that for a system of polynomials g_i there exists a set of vectors ξ_i linearly independent relative to M and such that the minimum polynomial associated with ξ_i is g_i and

$L_M(\xi_1, \xi_2, \cdots, \xi_k)$ is the whole space, is that the numbers $\sum\langle(h^k, g_i)\rangle$ are those specified by Theorem 27. This is sufficient because a diagonal block matrix N may be constructed with blocks of the form of the matrix in Theorem 11, having g_i as the minimum polynomial of each block.

The remainder of this section is devoted to describing a process which furnishes a method of testing whether two matrices are similar, and if M and N are similar, it furnishes a method of finding a set of polynomials g_i and sets of vectors ξ_i and η_i, each k in number, such that the ξ_i are linearly independent relative to M and the η_i are linearly independent relative to N; such that $L_M(\xi_1, \xi_2, \cdots, \xi_k) = L_N(\eta_1, \eta_2, \cdots, \eta_k) = S$, the total space; and such that g_i is minimally associated with ξ_i relative to M, and g_i is minimally associated with η_i relative to N. This process is wholly rational and does not involve the factorization of polynomials in the centrum into irreducible factors nor knowledge of a polynomial of minimum degree defining h; any factorization into powers of relatively prime factors in the centrum C of the minimum polynomial in C of M and N is suitable for an initial start and for each of these factors it is sufficient to have some polynomial g defining it. Either the required sets of vectors are found or one of the known factors of the minimum polynomial in C of M and N is factored with coefficients in C, or else a polynomial g of lower degree than the degree of the initial g is found which is associated with some factor of this minimum polynomial. The method therefore either yields the required vectors and polynomials or it reduces in a finite number of steps to the case studied in the beginning of this section. It should be borne in mind that this process is to be carried out simultaneously for M and N in practice, but the following treatment is for a matrix M only.

Let the minimum polynomial with coefficients in C for both M and N be $\prod h_i^{l_i}$, where the h_i are relatively prime. The whole vector space is the sum of the relative linear spaces orthogonal to $h_i^{l_i}$ and as above there is at least one vector such that $h_i^{l_i}$ is minimally associated with it. These spaces are linearly independent relative to M. To simplify notation, consider h to be any of the h_i, and g_i a transform of g_1. If, without assuming that the polynomials involved are irreducible, the method used in the beginning of this section is applied in constructing a proper base relative to a matrix M and a proper base relative to a matrix N, these bases having the properties described in the preceding paragraph, then the following situations may arise, and these are the only difficulties in applying this reasoning and process.

1. A vector ξ and a linear set U may be found such that $h^k(M) \odot \xi \neq 0 \bmod U$ and $h^{k+1}(M) \odot \xi = 0 \bmod U$, but the polynomial minimally associated with $\xi \bmod U$ over C is not h^{k+1}. This leads to a factorization of h.

2. g_1 and a set of transforms $g_2, g_3, \cdots, g_t$ exist such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ defines h^t but there exists a transform g_{t+1} such that $g_{t+1} \odot g_t \odot \cdots \odot g_1$ defines a multiple of h^t, not equal to h^t, and not equal to h^{t+1}. This also leads to a factorization of h.

3. A vector ξ and a linear set U exist such that $g_t \odot g_{t-1} \odot \cdots \odot g_1$ defines h^t, and h^t is minimally associated with ξ over C but ξ is minimally associated with a proper divisor, g_{11}, of $g_t \odot g_{t-1} \odot \cdots \odot g_1$. In this case the polynomial g_{11} defines h^t where $\langle g_{11} \rangle < t\langle g_1 \rangle$. By Theorem 19 (h, g_{11}) defines h. If $\langle (h, g_{11}) \rangle < \langle g_1 \rangle$ then one arrives immediately at a new start with a polynomial of lower degree than $\langle g_1 \rangle$ defining h. If $\langle (h, g_{11}) \rangle \geqq \langle g_1 \rangle$, let $g_{11} = g_{12} \odot (h, g_{11})$. By Theorem 16, g_{12} defines a power of h, less than or equal to the $(t-1)$st power, whereas $\langle g_{12} \rangle < (t-1)\langle g_1 \rangle$. Continuing with (h, g_{12}) defining h and by repeated use of the preceding argument, in a finite number of steps a polynomial is found, of degree lower than $\langle g_1 \rangle$ defining h. This case may arise in trying to find a vector ξ minimally associated with $g_t \odot g_{t-1} \odot \cdots \odot g_1$, and also if for such a ξ, $h^{t_1}(M) \odot \xi$ is minimally associated with a proper divisor of $g_{t-t_1} \odot g_{t-t_1-1} \odot \cdots \odot g_1$.

In carrying out this process simultaneously for M and N, one is led either to the proof of the non-similarity of M and N or the construction of proper bases for the whole space relative to M and N respectively, such that the polynomials minimally associated with the one base are equal to those minimally associated with the other.

This procedure which for the general case seems to be both long and involved is for such usual cases as third- and fourth-order matrices with quaternionic elements not very difficult to carry through. For practical operation it should be noted that if $\prod h_i^{l_i}$ is the minimum polynomial of M, and the h_i are relatively prime, then the columns of $\prod_{i \neq j} h_i^{l_i}$ form a base for the space orthogonal to $h_j^{l_j}(M)$.

It would seem that other problems could be advantageously attacked by methods of this paper. As an example, given two similar matrices M and N, all non-singular matrices S such that $S^{-1}MS = N$ consist of matrices S that transform $(\xi_i, M\xi_i, M^2\xi_i, \cdots, M^{\langle g_i \rangle - 1}\xi_i)$ into $(\eta_i, N\eta_i, N^2\eta_i, \cdots, N^{\langle g_i \rangle - 1}\eta_i)$, (i), where the ξ_i and η_i are sets of vectors whose relative linear extensions are the whole space and where g_i is minimally associated with ξ_i relative to M and with η_i relative to N. In particular, the non-singular matrices commutative with M are those which transform any set ξ_i into a set ξ_{1i}, having the same property.

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